ALLOWANCE FOR THE TRANSITION REGION IN THE CALCULATION OF PLANE AND AXISYMMETRIC BOUNDARY LAYERS

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Inzhenerno-Fizicheskii Zhurnal, Vol. 13, No. 1, pp. 9–15, 1967

UDC 532.526

An approximate method of allowing for the influence of the length of the transition region in the calculation of the characteristics of plane and axisymmetric boundary layers with a small pressure drop at the external boundary is proposed. A satisfactory agreement between the analytical results and experimental data is obtained.

In the proximity of the surface of a body moving in a liquid or a gas at large Reynolds numbers, there forms a boundary layer where laminar-turbulent transition occurs in a certain transition region. L. Prandtl [1] has schematized this region as a point and has proposed an approximate formula for determining the friction drag of a plate with allowance for the laminar and turbulent regions. A. R. Collar [2] has improved Prandtl's formula for a flat plate, within the limits of the concept of a transition point, while K. K. Fedyaevskii and V. T. Goroshchenko [3] have plotted curves for determining the profile drag of a wing. L. M. Zysina-Molozhen [4–6] was apparently the first to draw attention to the importance of taking into account the length of the transition region in the calculation of boundary layer characteristics and to propose an empirical computation method.

Let us examine the longitudinal flow past an axisymmetric body or a plane wing profile. The origin of the coordinates will be located at the forward stagnation point, the x axis will be directed along the contour, and the y axis will be normal to the contour. Applying the law for the change in momentum to an element of a boundary layer with a longitudinal pressure drop at the external boundary, we obtain the integral momentum relation [4, 7]

$$\frac{df}{dx} = \frac{dU}{dx}\frac{1}{U}F(f) + \left(\frac{d^2U}{dx^2}\Big/\frac{dU}{dx} - 2\frac{dr_0}{dx}\Big/r_0\right)f. \qquad (1)$$

Here, the following values are introduced:

$$F = (1+m)\zeta - [3+m+(1+m)H]f,$$

$$f = \frac{dU}{dx}\frac{\delta^{**}}{U}G(R^{**}),$$

$$H = \frac{\delta^*}{\delta^{**}}, \quad \zeta = \frac{\tau_0}{\rho U^2}G(R^{**}),$$

$$G(R^{**}) = \left(\frac{\rho U^2}{\tau_0}\right)_{f=0},$$

$$m(R^{**}) = \frac{d\log G(R^{**})}{d\log R^{**}}, \quad R^{**} = \frac{U\delta^{**}}{\nu},$$

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy,$$

$$\delta^* = \int_0^\delta \left(1-\frac{u}{U}\right)dy.$$

Relation (1), and all the following computations, are valid only for small values of δ/r_0. Without this limitation, the problem becomes appreciably more complicated, and a solution to it cannot be obtained in final form. In cases of practical importance, however, the transition region is usually located at the nose portion or at the maximum cross section of a body, where this limitation has only a slight effect.

It should be noted that (1) holds for the laminar and turbulent regions as well as for the transition region of an axisymmetric boundary layer. The term $2(dr_0/dx)(1/r_0)$ takes into account the influence of the transverse curvature of the body on the boundary layer characteristics. Postulating $r_0 \equiv 1$ in this relation and in all the following computations, we shall formally obtain formulas suitable for calculating the characteristics of a plane boundary layer.

Let us examine the transition region in a boundary layer, assuming that the dependence of the local friction coefficient in this region on the Reynolds number in the absence of a longitudinal pressure drop at the external boundary of the layer is governed by the power law

$$c_{f_{tr}} = \left(\frac{2\tau_0}{\rho U^2}\right)_{f=0} = AR_x^B. \qquad (2)$$

The values of the constants A and B will be determined from the condition that the value of the coefficient $c_{f_{tr}}$ at the beginning of the transition region is equal to its value at the end of the laminar region, while its value at the end of the transition region is equal to that at the beginning of the turbulent region; i.e.,

$$c_{f_{tr}} = c_{f_l} \quad \text{for} \quad R_x = R_{x_l}, \qquad c_{f_{tr}} = c_{f_t} \quad \text{for} \quad R_x = R_{x_T}. \qquad (3)$$

The local friction coefficient in the laminar and turbulent regions of the boundary layer will be calculated on the basis of the Blasius [10] and Falkner [11] formulas, respectively:

$$c_{f_l} = \frac{0.664}{R_x^{1/2}}; \quad c_{f_t} = \frac{0.0263}{R_x^{1/7}}. \qquad (4)$$

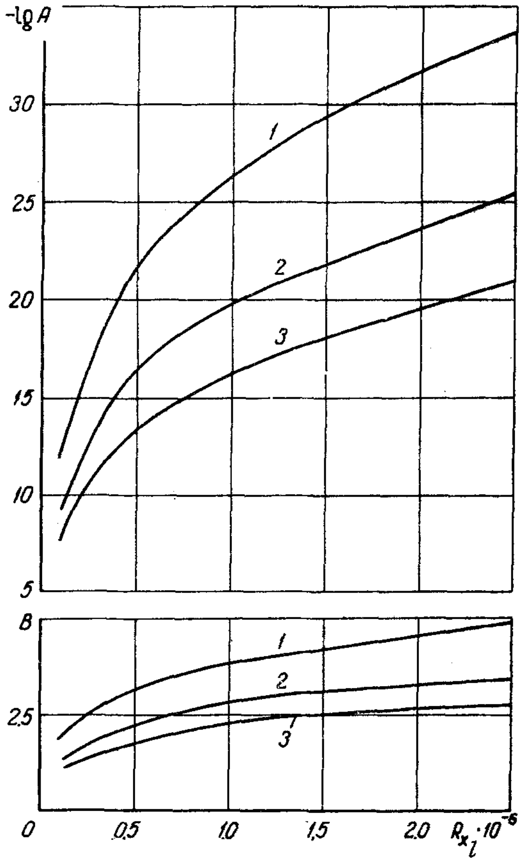

Fig. 1. Coefficients A and B plotted vs. the critical Reynolds number at the beginning of the transition region. 1) k = = 1.5; 2) k = 1.75; 3) k = 2.0.

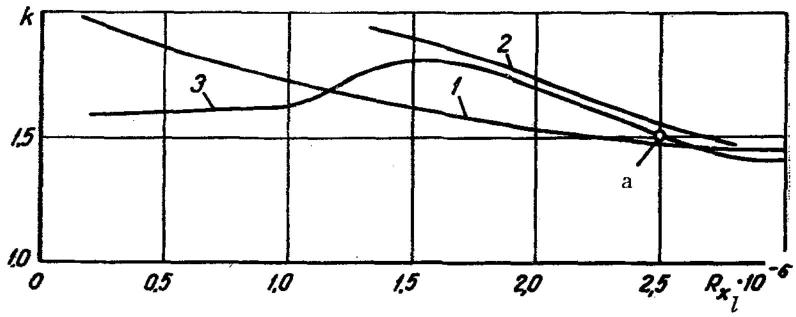

Fig. 2. The coefficient $k = R_{x_t}/R_{x_l}$ as a function of the critical Reynolds number at the beginning of the transition region: 1) according to the data of [12]; 2) according to [13]; 3) according to [6]; a corresponds to [14].

Then, after simple computations we obtain

$$B=\frac{\log\dfrac{0.0263}{0.664}}{\log k}+\frac{5}{14}\,\frac{\log R_{x_l}}{\log k}-\frac{1}{7}\,; \tag{5}$$

$$A=0.664\;R_{x_l}^{-(B+1/2)}, \tag{6}$$

where $k = R_{x_t}/R_{x_l}$.

Figure 1 shows the coefficients A and B as a function of the critical Reynolds number at the beginning of the transition region R_{x_l}.

It should be noted that, using a somewhat different approach, Zysina-Molozhen [6] has derived an analogous coefficient $r = x_t/x_l$, which in the case of a plate is identical to the coefficient $k = R_{x_t}/R_{x_l}$. A method of calculating R_{x_l} with allowance for the initial turbulence of the flow and surface roughness of the body is outlined in [9].

Processing of the experimental data obtained by Potter and Whitfield [12], Schubauer and Skramstad [13], and Schubauer and Klebanoff [14] concerning the length of the transition region in the boundary layer on a plate made it possible to obtain a plot of the coefficient k vs. the critical Reynolds number R_{x_l}, shown in Fig. 2. Curve (1) is plotted on the basis of the data obtained by Potter-Whitfield [12], who have generalized the experimental results. Curve (2) is a result of the processing of Schubauer and Skramstad's [13] experimental data. Curve (3) is obtained by conversion of a similar graph published by Zysina-Molozhen in [6]. Point a in the graph corresponds to the experimental results obtained by Schubauer and Klebanoff [14]. The value of the coefficient k computed from Dhawan and Narasimha's [15] formula appears to exceed the experimental data by 15–20%.

The specific nature of the relation shown in Fig. 3 should be emphasized, since in the general case, coefficient k is not a single-valued function of the local critical Reynolds number R_{x_l}, which depends not only on the degree of turbulence of the oncoming flow but also on the longitudinal pressure drop at the external boundary of the layer. Analysis of the experimental data [16, 17] for a boundary layer with a small pressure drop at the external boundary shows, however, that in this case the values of the coefficient k correlate satisfactorily with those shown in Fig. 2 for a plate.

In order to determine the relationship between the local friction coefficient $c_{f_{tr}}$ and the Reynolds number R^{**} in the transition region, we shall use the integral relation (1), which for the particular case of a plate is written in the form

$$\frac{1}{2}\,c_{f_{tr}}=\frac{dR^{**}}{dR_x}=\frac{1}{2}\,AR_x^B=\frac{1}{G(R^{**})}\,. \tag{7}$$

After separating the variables, we shall integrate the relation (7) for the boundary condition $R^{**} = 0$ for $R_x = 0$. As a result, we get

$$R_x^B=\left(\frac{B+1}{A/2}\right)^{B/(B+1)}R^{**B/(B+1)}\,. \tag{8}$$

With allowance for the relations (7) and (8), we shall calculate the normalizing function

$$G(R^{**})=\frac{2}{A}\left(\frac{B+1}{A/2}\right)^{-B/(B+1)}R^{**-B/(B+1)}\,. \tag{9}$$

After taking the logarithm of the normalizing function and subsequently differentiating it, we get

$$m=\frac{d\log G(R^{**})}{d\log R^{**}}=-\frac{B}{B+1}\,. \tag{10}$$

Processing of experimental data for the transition region [14, 18] showed that for small pressure drops at the external boundary of the layer—for which all the following results are valid—one may postulate $\zeta \cong 1$ and $H \cong 2$ in relation (1). For a boundary layer with a large longitudinal pressure gradient at the external boundary, the value of the parameter H varies within the range from 2.0 to 3.6, as has been correctly pointed out by Zysina-Molozhen [4]. It is, therefore, advisable to use the mean value H = 3 in such cases.

For the case of a boundary layer with a small pressure drop at the external boundary, the function $F(f)$ becomes linear and reduces to the form

$$F(f)=\frac{1}{B+1}-\frac{(5+2B)}{B+1}\,f, \tag{11}$$

while relation (1) becomes a Bernoulli differential equation with an integral of the following form:

$$f(x)=\frac{dU}{dx}\,\frac{1}{U^{\frac{5+2B}{B+1}}(x)\,r_0^2(x)}\times$$
$$\times\left(\frac{1}{B+1}\int_{x_l}^{x}U^{\frac{4+B}{B+1}}(\xi)\,r_0^2(\xi)\,d\xi+\right.$$
$$\left.+\,f_l r_{0_l}^2 U_l^{\frac{5+2B}{B+1}}\,\frac{1}{dU_l/dx}\right). \tag{12}$$

Here, the values of the functions with the subscript l constitute the corresponding values at the end of the laminar portion of the boundary layer, the formula

$$f_l=4.54\,R_l^{**^2}\,\frac{\nu}{U_l^2}\,\frac{dU_l}{dx} \tag{13}$$

being recommended for calculating the value of f_l. The values of R_l^{**} can be determined from laminar boundary layer theory [8].

After having calculated the form parameter $f(x)$, R_x^{**} or $\delta^{**}(x)$ can be determined from the equation

$$\frac{2}{A}\left(\frac{B+1}{A/2}\right)^{-B/(B+1)}R^{**1/(B+1)}=fU^2\Big/\nu\,\frac{dU}{dx}\,. \tag{14}$$

Having determined $R^{**}(x)$ and $R_x(x)$, the ratios of the relative boundary layer thicknesses in the transition region are calculated from formula

$$H_{tr}(x)=H_l-\left(1-\frac{R_x}{R_{x_l}}\right)\left(\frac{H_l-H_t}{1-R_{x_t}/R_{x_l}}\right), \tag{15}$$

where $H_l = H(x_l)$ and $H_t = H(x_t)$.

This formula was obtained under the assumption that H varies linearly with R_x. The value of H_l can be determined analytically (see monograph [8]). For boundary layer calculations involving a small pressure drop at the external boundary, it is advisable to use $H_t = 1.3-1.4$. A comparison of the results obtained from the formula with Schubauer and Klebanoff's [14] experimental data showed satisfactory agreement within the limits of from 5 to 8%.

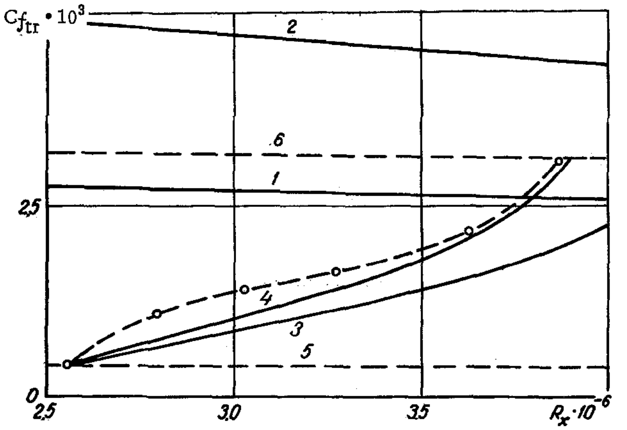

Fig. 3. Comparison of the analytical and experimental plots of the local friction coefficient vs. Reynolds number. 1) and 2), according to Zysina-Molozhen [6] for x = 0.30 and 0.41, respectively; 3) according to Dhawan-Narasimha [15]; 4) the method proposed; 5) from the Blasius formula [10]; 6) from Falkner's formula [11]; the points correspond to experimental values [14].

Further, we shall calculate the displacement thickness

$$\delta^*(x) = H\,\delta^{**}(x) \tag{16}$$

and the local shear stress

$$\frac{\tau_0}{\frac{1}{2}\rho U^2} = A\left(\frac{B+1}{A/2}\right)^{B/(B+1)} R^{**B/(B+1)}. \tag{17}$$

Within the limits of the assumption employed, formula (17) may be recommended for appraisal computations.

A comparison of analytical and experimental data for the local friction coefficient is given in Figure 3. The experimental points in the figure are obtained by processing the velocity profiles measured by Schubauer and Klebanoff [14]. The figure also shows the corresponding theoretical values calculated on the basis of Zysina-Molozhen's data (curves 1 and 2), Dhawan-Narasimha's [15] data (curve 3), and by the method proposed (curve 4), as well as on the basis of the formulas proposed by Blasius [10] (curve 5) and by Falkner [11] (curve 6). It should be emphasized that curve 1 is obtained from calculations performed on the basis of formula (9) in [6] (p. 454) for x = 0.30, while curve 2 is obtained for x = 0.41. The range of variation of x has been selected in correspondence with the data taken from Zysina-Molozhen. The data presented demonstrate the satisfactory agreement between values obtained by the method proposed and experimental values.

NOTATION

r_0 is the instantaneous radius of the axisymmetric body; δ is the boundary layer thickness; δ^{**} is the momentum thickness; δ^* is the displacement thickness; x_l is the abscissa of the terminal point of the laminar region; x_t is the abscissa of the beginning of the turbulent region; u is the projection of the boundary layer velocity vector on the x axis; U is the velocity at the external boundary of the boundary layer; f is the form parameter of the boundary layer; ζ is the dimensionless local friction coefficient; H is the ratio of the relative boundary layer thicknesses; ν and μ are the kinematic and dynamic viscosity coefficients of the fluid, respectively; ρ is the density of the fluid; τ_0 is the shear stress at the surface of the body; $G(R^{**})$ is the normalizing function; $R_x = Ux/\nu$; $R^{**} = U\delta^{**}/\nu$ are the local Reynolds numbers; $R_{xl} = Ux_l/\nu$; $R_{xt} = Ux_t/\nu$ are the local critical Reynolds numbers.

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10 November 1966
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